Non-polynomial Third Order Equations which Pass the Painlevé Test

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The singular point analysis of third-order ordinary differential equations in the non-polynomial class is presented. Some new third order ordinary differential equations which pass the Painlevé test, as well as the known ones are found.

Key words: Painlevé Equations, Painlevé Test.

1. Introduction

Painlevé and his school [1 – 3] studied a certain class of second order ordinary differential equations (ODE's) and found fifty canonical equations whose solutions have no movable critical points. This property is known as the Painlevé property. Distinguished among these fifty equations are six Painlevé equations, PI-PVI, which are regarded as nonlinear special functions.

The third order Painlevé type equations

$$y''' = F(z,y,y',y''), \tag{1.1}$$

where F is polynomial in y and its derivatives, were considered in [4 – 7]. Some fourth and higher order polynomial-type equations with the Painlevé property were investigated in [5 – 10].

The third order equation (1.1), such that F is analytic in z and rational in its other arguments, was considered in [11, 12]. [12] starts with a simplified equation. i.e. an equation which contains terms with leading order $\alpha = -1$ as $z \to z_0$ only:

$$y''' = \left(1-\frac{1}{\nu}\right)\frac{(y''-2yy'^2)}{y'-y^2} + c_1\frac{y'y''}{y} + c_2\frac{(y')^3}{y^2} + a_1yy'' + a_2(y')^2 + a_3y^2y' + a_4y^4, \tag{1.2}$$

where $a_i =$ constant, $i = 1,2,3,4$, $\nu \in \mathbb{Z} - \{-1,0\}$, $c_j =$ constant, $j = 1,2$, $c_1^2 + c_2^2 \neq 0$, and investigates the values of a_i and c_j such that the equation is of Painlevé type.

We consider the third order differential equation

$$y''' = c_1\frac{y'y''}{y} + c_2\frac{(y')^3}{y^2} + F(y,y',y'';z), \tag{1.3}$$

where c_1 and c_2 are constants, such that $c_1^2 + c_2^2 \neq 0$. F may contain the leading terms, but all the terms of F are of order ε^{-2} or greater if we let $z = z_0 + \varepsilon t$, where ε is a small parameter, t is the new independent variable and the coefficients of F are locally analytic functions of z. The equation of type (1.3) can be obtained by differentiating the leading terms of the third Painlevé equation and adding the terms of order -4 or greater as $z \to z_0$ with the analytic coefficients in z such that: **i.)** $y = 0, \infty$ are the only singular values of the equation in y, **ii.)** The additional terms are of order ε^{-3} or greater, if one lets $z = z_0 + \varepsilon t$.

If we let $z = z_0 + \varepsilon t$ and take the limit as $\varepsilon \to 0$, (1.3) yields the "reduced" equation

$$\dddot{y} = c_1\frac{\dot{y}\ddot{y}}{y} + c_2\frac{\dot{y}^3}{y^2}, \tag{1.4}$$

where $\dot{} = d/dt$. Substituting $y \cong y_0(t-t_0)^\alpha$ into (1.4) gives

$$(c_1 + c_2 - 1)\alpha^2 - (c_1 - 3)\alpha - 2 = 0. \tag{1.5}$$

Let $c_1 + c_2 - 1 \neq 0$ and the roots of (1.5) be $\alpha_1 = n$ and $\alpha_2 = m$ such that $n, m \in \mathbb{Z} - \{0\}$, then

$$(1-m-n)c_1 - (n+m)c_2 + m + n - 3 = 0, \tag{1.6}$$

$$(n-m)^2(c_1+c_2-1)^2 - c_1(c_1+2) - 8c_2 - 1 = 0.$$

If $n+m-1 \neq 0$, then

$$(c_2+2)[2(1-m-n+mn)+mnc_2] = 0. \quad (1.7)$$

It should be noted that if $c_2 = -2$, then $c_1 = 3$ and $c_1+c_2-1 = 0$. So we have

$$(c_1, c_2) = \Big(\frac{1}{mn}(3mn-2n-2m), \frac{2}{mn}(m+n-mn-1)\Big) \quad (1.8)$$

when $n+m-1 \neq 0$, $c_1 \neq 3$ and $c_1+c_2-1 \neq 0$.

Substituting

$$y \cong y_0(t-t_0)^{\alpha} + \beta(t-t_0)^{r+\alpha} \quad (1.9)$$

into (1.4), we obtain the equations for the Fuchs indices in the form

$$\begin{aligned} &r(r+1)[mr+2(n-m)] = 0, \text{ and} \\ &r(r+1)[nr-2(n-m)] = 0 \end{aligned} \quad (1.10)$$

for $\alpha = n$ and $\alpha = m$, respectively. So, the Fuchs indices are,

$$\begin{aligned} (r_0, r_1, r_2) &= \Big(-1, 0, 2-\frac{2n}{m}\Big), \\ (r_0, r_1, r_2) &= \Big(-1, 0, 2-\frac{2m}{n}\Big) \end{aligned} \quad (1.11)$$

for $\alpha = n$ and $\alpha = m$ respectively. In order to have distinct indices, if $p = 2n/m$, $q = 2m/n$ than $p, q \in \mathbb{Z}$ and satisfy the Diophantine equation $pq = 4$. By solving the Diophantine equation for p, q and using the symmetry of (1.8) with respect to n and m, one gets the following 3 cases for (c_1, c_2):

$$\begin{aligned} &\mathbf{1.} \quad (c_1, c_2) = \Big(3, -2+\frac{2}{n^2}\Big), \\ &\mathbf{2.} \quad (c_1, c_2) = \Big(3-\frac{1}{n}, -2+\frac{1}{n}+\frac{1}{n^2}\Big), \\ &\mathbf{3.} \quad (c_1, c_2) = \Big(3-\frac{3}{n}, -2+\frac{3}{n}-\frac{1}{n^2}\Big). \end{aligned} \quad (1.12.\text{a,b,c})$$

If $n+m-1 = 0$, (1.6) and $c_1+c_2-1 \neq 0$ imply that $c_2 = -2$ and $c_1 \neq 3$, respectively. Then

$$\begin{aligned} (c_1, c_2) &= \Big(\frac{3n^2-3n+2}{n(n-1)}, -2\Big), \\ &n \neq 0, 1, \text{ and } c_1 \neq 3. \end{aligned} \quad (1.13)$$

Similarly, substituting (1.9) into (1.4) with the values of (c_1, c_2) given in (1.13) gives the equations for the Fuchs indices in the form

$$\begin{aligned} &r(r+1)[r(n-1)+2(1-2n)] = 0, \text{ and} \\ &r(r+1)[nr-2(1-2n)] = 0 \end{aligned} \quad (1.14)$$

for $\alpha = n$ and $\alpha = m = 1-n$, respectively. In order to have distinct Fuchs indices for both branches $\alpha = n$ and $\alpha = m$, n must take the values $-1, 2$. Therefore, when $n+m-1 = 0$ and $c_1+c_2-1 \neq 0$ we have $(c_1, c_2) = (4, -2)$, which can be obtained from (1.12b) for $n = -1$.

In the case of the single branch, i.e. $c_1+c_2-1 = 0$, let $\alpha = n \in \mathbb{Z}-\{0\}$, the Fuchs indices are $r = -1, 0, 2$, and the coefficients (c_1, c_2) are

$$\mathbf{4.} \quad (c_1, c_2) = \Big(3-\frac{2}{n}, -2+\frac{2}{n}\Big). \quad (1.15)$$

If $c_1+c_2-1 = 0$ and $c_1 = 3$, then $c_2 = -2$. So, as the fifth case we have

$$\mathbf{5.} \quad (c_1, c_2) = (3, -2). \quad (1.16)$$

Thus, we have five cases, (1.12), (1.15) and (1.16), and all the corresponding equations pass the Painlevé test. Moreover, if one lets $y = u^n$ in (1.4) with the coefficients (c_1, c_2) given by (1.12) and (1.15), and integrates the resulting equation for u once, then u satisfies a linear equation or is solvable by means of elliptic functions. For (c_1, c_2) given by (1.16), (1.4) yields $\ddot{u} = 0$ if we let $u = \dot{y}/y$ and integrate the resulting equation twice. Therefore all five equations have the Painlevé property.

2. Leading Order $\alpha = -1$

Equation (1.4) contains the leading terms for any $\alpha \in \mathbb{Z}-\{0\}$ as $z \to z_0$. In this section we consider the case $\alpha = -1$. By adding the terms of order -4 or greater as $z \to z_0$, we obtain the equation

$$\begin{aligned} y''' = c_1\frac{y'y''}{y} + c_2\frac{(y')^3}{y^2} + a_1yy'' + a_2(y')^2 \\ + a_3y^2y' + a_4y^4 + F_j(y, y', y'', z), \end{aligned} \quad (2.1)$$

where a_i, $i = 1, \ldots, 4$ are constants and F_j, $j = 1, 2$:

$$\begin{aligned} F_1 = A_1y'' + A_2\frac{(y')^2}{y} + A_3yy' + A_4y^3 + A_5\frac{y''}{y} \\ + A_6y' + A_7y^2 + A_8\frac{y'}{y} + A_9y + A_{10} + A_{11}\frac{1}{y}, \end{aligned}$$

$$\begin{aligned} F_2 = A_1y'' + A_2\frac{(y')^2}{y} + A_3yy' + A_4y^3 + A_5\frac{y''}{y} \\ + A_6\Big(\frac{y'}{y}\Big)^2 + A_7y' + A_8y^2 + A_9\frac{y''}{y^2} + A_{10}\frac{y'}{y} + A_{11}y \end{aligned}$$

$$+A_{12}\frac{y'}{y^2}+A_{13}+A_{14}\frac{1}{y}+A_{15}\frac{1}{y^2}, \qquad (2.2.a,b)$$

if $c_2=0$ and $c_2\neq 0$, respectively, and where $A_k(z)$ are locally analytic functions of z. (2.1) contains all the leading terms for $\alpha=-1$, if we do not take into account F_j.

Suppose that (1.12), (1.15) and (1.16) hold and substitute [13]

$$y\cong y_0(z-z_0)^{-1}+\beta(z-z_0)^{r-1} \qquad (2.3)$$

into (2.1) without F_1. Then we obtain the following equations for the Fuchs indices (resonances) r and y_0

$$Q(r)=(r+1)[r^2-(a_1y_0+7-c_1)r+3(6-2c_1-c_2)$$
$$+2(2a_1+a_2)y_0-a_3y_0^2]=0, (2.4.a,b)$$
$$a_4y_0^3-a_3y_0^2+(2a_1+a_2)y_0+6-2c_1-c_2=0,$$

respectively. Equation (2.4.b) implies that, in general, there are three branches if $a_4\neq 0$. Now we determine $y_{0j},\ j=1,2,3,$ and $a_i,\ i=1,2,3,4,$ for each case of (c_1,c_2) such that at least one branch is the principal branch, i.e. all the resonances are positive and distinct integers (except $r_0=-1$). A_k can be determined by using the transformation

$$y=\mu(z)\tilde{y}(x), \quad x=\rho(z), \qquad (2.5)$$

which preserves the Painlevé property, where μ and ρ are locally analytic functions of z and the compatibility conditions at the Fuchs indices r_{ji} and the compatibility conditions corresponding to parametric zeros; that is, the compatibility conditions at the Fuchs indices $\tilde{r}_{ji}$ of the equations obtained by the transformation $y=1/u$.

According to the number of branches, the following cases should be considered separately.

Case I. $a_3=a_4=0$: In this case there is one branch. If $r_0=-1$ and (r_1,r_2) are resonances, then (2.4.b) implies that

$$-(2a_1+a_2)y_0=r_1r_2=6-2c_1-c_2,$$
$$r_1+r_2=a_1y_0+7-c_1. \qquad (2.6)$$

In order to have a principal branch,

$$6-2c_1-c_2=k, \quad k\in\mathbb{Z}_+. \qquad (2.7)$$

When $(c_1,c_2)=(3,-2+\frac{2}{n^2})$, (2.7) implies that $n=\pm1$. Then $y_0\neq 0$ and arbitrary, the Fuchs indices are $(r_1,r_2)=(0,4)$ and the simplified equation is [12]

$$y'''=3\frac{y'y''}{y}. \qquad (2.8)$$

Integrating (2.8) once yields $y''=k_1y^3$, where k_1 is an integration constant.

In this case, the canonical form of the equation is

$$y'''=3\frac{y'y''}{y}+A_1y''+A_2\frac{(y')^2}{y}+A_3yy'+A_4y^3+A_5\frac{y''}{y}$$
$$+A_6y'+A_7y^2+A_8\frac{y'}{y}+A_9y+A_{10}+A_{11}\frac{1}{y}. \qquad (2.9)$$

$A_3=0$, otherwise $\alpha=-2$ is of leading order. The transformation (2.5) allows one to take $A_1=A_2=0$. If we substitute

$$y=(z-z_0)^{-1}+\sum_{i=0}^{\infty}y_i(z-z_0)^i \qquad (2.10)$$

in (2.9), then the compatibility condition at $r_2=4$ gives that $A_4=A_7=0$ and

$$A_5''+A_{10}-A_8'=0,\ A_6''-2A_9'=0. \qquad (2.11)$$

If we let $y=1/u$, then (2.9) yields

$$uu''=3u'u''+A_5\left[u^2u''-2u(u')^2\right]+A_6uu'$$
$$+A_8u^2u'-A_9u^2-A_{10}u^3-A_{11}u^4. \qquad (2.12)$$

$\tilde{\alpha}=-1$ is the possible leading order of u as $z\to z_0$, if $A_5=A_{11}=0$ and the Fuchs indices are $(\tilde{r}_1,\tilde{r}_2)=(0,4)$. The compatibility condition at $\tilde{r}_2=4$ together with (2.11) gives $A_8=k_1=$ constant, $A_{10}=0$, $A_9'=A_6''=0$ and

$$k_1(A_6'+2A_9)=0. \qquad (2.13)$$

If $k_1=0$, then the canonical form of the equation is

$$yy'''=3y'y''+(k_2z+k_3)yy'+k_4y^2. \qquad (2.14)$$

If one lets $y=e^v$ and $v'=w$, then (2.14) yields the second Painlevé equation. If $k_1\neq 0$, then we have

$$yy'''=3y'y''-(2k_2z-k_3)yy'+k_1y'+k_2y^2, \qquad (2.15)$$

where $k_i=$ constant for $i=2,3$. Integrating (2.15) once yields

$$y''=k_4y^3+\frac{1}{2}(2k_2z-k_3)y-\frac{k_1}{3},\ k_4=\text{constant}. \qquad (2.16)$$

(2.16) is of Painlevé type [14].

When $(c_1, c_2) = (3 - 1/n, -2 + 1/n + 1/n^2)$, (2.7) implies that $n = \pm 1$. For $n = -1$, $y_0 =$ arbitrary $\neq 0$, $r_2 = 3$, $(r_1, r_2) = (0, 3)$

$$y''' = 4\frac{y'y''}{y} - 2\frac{(y')^3}{y^2}. \tag{2.17}$$

Integration of (2.17) once yields

$$y'' = \frac{1}{2}\frac{(y')^2}{y} + k_1 y^3, \; k_1 = \text{constant}, \tag{2.18}$$

which is solvable by means of elliptic functions.

After adding the non-dominant terms F_2 given by (2.2.b), the leading order is $\alpha = -1$ if $A_3 = 0$. The compatibility condition at $r_2 = 3$ implies that $A_5 = A_6 = 0$. On the other hand, if $A_9 = 0$, then the leading order of $u = 1/y$ as $z \to z_0$ is $\tilde{\alpha} = -1$. The following two cases may be considered separately:

If $A_{12} \neq 0$ and $A_{15} = 0$, then $A_{12}(z_0)u_0^2 = 2$ and the Fuchs indices of u are $(\tilde{r}_{j1}, \tilde{r}_{j2}) = (1, 4)$, $j = 1, 2$. The compatibility conditions at $\tilde{r}_{ji}$ of both branches of u, together with the compatibility condition at r_2, give that $A_k = 0$ for all k except $A_7 = k_1$, $A_8 = k_2$, $A_{12} = k_3$, $k_i =$ constant, $i = 1, 2, 3$. Then we obtain the equation

$$y^2 y''' = 4yy'y'' - 2(y')^3 + k_1 y^2 y' + k_2 y^4 + k_3 y'. \tag{2.19}$$

If $A_{15} \neq 0$ and $A_{12} = 0$, then $A_{15}(z_0)u_0^3 = -2$, $(\tilde{r}_{j1}, \tilde{r}_{j2}) = (2, 3)$, $j = 1, 2, 3$. The compatibility conditions at $\tilde{r}_{ji}$ of all the three branches of u together with the compatibility condition at r_2 give that $A_8 = k_1$, $A_{15} = k_2$, k_i=constant, $i = 1, 2$, and the rest of the coefficients $A_k = 0$. Then we have

$$y^2 y''' = 4yy'y'' - 2(y')^3 + k_1 y^4 + k_2. \tag{2.20}$$

For $n = 1$, the Fuchs indices and the simplified equation are

$$y_0 = -\frac{2}{a_1}: \quad (r_1, r_2) = (1, 2), \qquad y''' = 2\frac{y'y''}{y} + a_1[yy'' - (y')^2]. \tag{2.21.a,b}$$

Equation (2.21.b) does not pass the Painlevé test, since the compatibility condition at $r_2 = 2$ is not satisfied identically. (2.21.b) was considered in [12].

When $(c_1, c_2) = (3 - \frac{3}{n}, -2 + \frac{3}{n} - \frac{1}{n^2})$, (2.7) implies that $n = \pm 1$. For $n = 1$, $(c_1, c_2) = (0, 0)$. This case leads to a polynomial type equation. For $n = -1$, let $r_1 = 0$, then $y_0 =$ arbitrary $\neq 0$, $r_2 = 1$ and

$$y''' = 6\frac{y'y''}{y} - 6\frac{(y')^3}{y^2}. \tag{2.22}$$

If we let $y = 1/u$, then (2.22) yields $u''' = 0$. Equation (2.22) was considered in [12].

By adding the non-dominant terms F_2 and applying the same procedure, we obtain the following canonical form of the equations: If $A_9 = A_{15} = 0$ and $A_{12}(z) \neq 0$, then $u = (1/y) \sim (z - z_0)^{-1}$ as $z \to z_0$ $(\tilde{\alpha} = -1)$, the Fuchs indices are $(\tilde{r}_1, \tilde{r}_2) = (3, 4)$ and the canonical form of the equation is

$$\begin{aligned} y^2 y''' &= 6yy'y'' - 6(y')^3 + 4(z^2 + k_1)y^2 y' \\ &\quad + 12zyy' - 4zy^3 + 6y' + 4y^2, \end{aligned} \tag{2.23}$$

where k_1 is a constant. If $A_9 = A_{12} = A_{14} = A_{15} = 0$ and $A_{10}(z) \neq 0$, then $\tilde{\alpha} = -2$, $(\tilde{r}_1, \tilde{r}_2) = (4, 6)$ [6], and the canonical form of the equation is

$$\begin{aligned} y^2 y''' &= 6yy'y'' - 6(y')^3 + \frac{1}{z}\left[y^2 y'' - 2y(y')^2\right] \\ &\quad + \left(\frac{1}{z} + 6z^3\right)y^4 + 12yy - 12zy^3 + \frac{6}{z}y^2 \\ &= 6yy'y'' - 6(y')^3 + \left(\frac{k_1^2 z}{6} - k_2\right)y^4 - k_1 y^3 + 12yy, \end{aligned} \tag{2.24}$$

where k_1 and k_2 are constants. If $A_9 = A_{10} = A_{12} = A_{13} = A_{14} = A_{15} = 0$, then u satisfies a linear equation and the canonical form of (2.24) is

$$\begin{aligned} y^2 y''' &= 6yy'y'' - 6(y')^3 + A_1[y^2 y'' - 2y(y')^2] \\ &\quad + A_7 y^2 y' + A_8 y^4 + A_{11} y^3, \end{aligned} \tag{2.25}$$

where A_1, A_7, A_8, A_{11} are arbitrary locally analytic functions of z.

When $(c_1, c_2) = (3 - 2/n, -2 + 2/n)$, (2.7) implies that $n = \pm 1, \pm 2$. For $n = -1$, let $r_1 = 0$, then $y_0 =$ arbitrary $\neq 0$, $r_2 = 2$ and the simplified equation is

$$y''' = 5\frac{y'y''}{y} - 4\frac{(y')^3}{y^2}. \tag{2.26}$$

Integration of (2.26) once yields

$$y'' = \frac{(y')^2}{y} + k_1 y^3, \quad k_1 = \text{constant}, \tag{2.27}$$

which is solvable by means of elliptic functions.

If we add the non-dominant terms F_2 given in (2.2.b) to (2.26) then we should set $A_3 = 0$, in order to have the leading order $\alpha = -1$. The transformation (2.5) and the compatibility condition at $r_2 = 2$ imply that $A_5 = A_6 = 0$ and $A_4 = A_8 = 0$, respectively. On the other hand, if $A_9 = A_{15} = 0$ and $A_{12} \neq 0$, then $\tilde{\alpha} = -1$, $A_{12}(z_0)u_0^2 = 4$ and $(\tilde{r}_{j1}, \tilde{r}_{j2}) = (2,4)$, $j = 1,2$. The compatibility conditions at $\tilde{r}_{ij}$ imply that all the coefficients A_k are zero except $A_{10} = k_1$ and $A_{12} = k_2$, $k_i =$ constant, $i = 1,2$. Therefore, the canonical form of the equation is

$$y^2y''' = 5yy'y'' - 4(y')^3 + k_1yy' + k_2y'. \quad (2.28)$$

For $n = 1$, (2.6) implies that $r_1r_2 = 4$. Then the Fuchs indices and the simplified equation are

$$y_0 = -\frac{1}{a_1}: \quad (r_1, r_2) = (1,4), \qquad y''' = \frac{y'y''}{y} + a_1\left[yy'' + 2(y')^2\right]. \quad (2.29.a,b)$$

(2.29) was also considered in [12]. If one replaces y by λy such that $a_1\lambda = -1$ and lets $y = 1/u$, (2.29.b) yields

$$u^2u''' = 5uu'u'' - 4(u')^3 - uu'' + 4(u')^2. \quad (2.30)$$

(2.30) does not pass the Painlevé test. Hence (2.30), and consequently (2.29) is not of Painlevé type.

For $n = 2$, y_0, the Fuchs indices and the simplified equation are

$$y_0 = -\frac{1}{a_1}: \quad (r_1, r_2) = (1,3), \qquad y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + a_1\left[yy'' + (y')^2\right], \quad (2.31.a,b)$$

respectively. (2.31.b) was considered in [12], and its first integral is

$$y'' = \frac{(y')^2}{y} + a_1yy' + k_1, \quad k_1 = \text{constant}. \quad (2.32)$$

(2.32) is of Painlevé type [3, 14].

If we add the non-dominant terms to (2.31), the leading order $\tilde{\alpha}$ depending on of u as $z \to z_0$, we have the following canonical form of the equations: If $\tilde{\alpha} = -1$ then $(\tilde{r}_{j1}, \tilde{r}_{j2}) = (1,2)$, $j = 1,2$. The transformation (2.5), the compatibility conditions at $\tilde{r}_{ji}$, $i, j = 1,2$, and the compatibility conditions at $(r_1, r_2) = (1,3)$ are enough to determine all the coefficients A_k in terms of A_1. Then, one gets the following canonical form of the equation

$$\begin{aligned} y^2y''' &= 2yy'y'' - (y')^3 - y^3y'' - y^2(y')^2 \\ &+ A_1\left[y^2y'' - y(y')^2 + y^3y'\right] + A_1'y^2y' \\ &+ (A_1'' - A_1A_1')y^3 + A_{12}(y' + y^2) - A_1A_{12}y, \end{aligned} \quad (2.33)$$

where $A_{12}' = 2A_1A_{12}$.

If $\tilde{\alpha} = -2$, then $A_5 = A_6 = A_{10} = A_{12} = A_{15} = 0$, and $(\tilde{r}_1, \tilde{r}_2) = (0,2)$. The compatibility condition at $\tilde{r}_2 = 2$ gives that $A_8 = A_{13} = 0$ and $A_7 = A_1'$, $A_{11} = A_1'' - A_1'A_1$. Then, the canonical form of the equation is

$$\begin{aligned} y^2y''' &= 2yy'y'' - (y')^3 - y^3y'' - y^2(y')^2 \\ &+ A_1\left[y^2y'' - y(y')^2 + y^3y'\right] + A_1'y^2y' \\ &+ (A_1'' - A_1'A_1)y^3, \end{aligned} \quad (2.34)$$

where A_1 is a locally analytic function of z.

For $n = -2$, since $r_1r_2 = 1$, $r = \pm 1$ are the double Fuchs indices.

When $(c_1, c_2) = (3, -2)$, y_0, the Fuchs indices and the simplified equation are

$$y_0 = -\frac{1}{a_1}: \quad (r_1, r_2) = (1,2), \qquad y''' = 3\frac{y'y''}{y} - 2\frac{(y')^3}{y^2} + a_1yy''. \quad (2.35)$$

(2.35) was also considered in [12].

If one adds the non-dominant terms, then $\tilde{\alpha} = -1$ when $A_6 = -2A_5$, $A_9 = A_{12} = A_{15} = 0$ and $A_5(z_0)u_0 = -1$, $(\tilde{r}_1, \tilde{r}_2) = (1,2)$. Therefore, the canonical form of the equation is

$$\begin{aligned} y^2y''' &= 3yy'y'' - 2(y')^3 - y^3y' \\ &+ A_1\left[y^2y'' - 2y(y')^2 - y^3y' - y^5\right] \\ &+ A_5\left[yy'' - 2(y')^2\right] + A_7(y^2y' + y^4) \\ &+ (2A_5' - 3A_1A_5)yy' + A_{11}y^3 \\ &- (A_5'' - A_1A_5' - A_5A_7)y^2 - A_1A_5^2y, \end{aligned} \quad (2.36)$$

where A_1, A_5, A_7 and A_{11} are arbitrary locally analytic functions of z.

Case II. $a_3 \neq 0$, $a_4 = 0$: If y_{0j}, $j = 1,2$, are roots of (2.4.b), and (r_{j1}, r_{j2}) are the Fuchs indices corresponding to y_{0j}, then let

$$r_{j1}r_{j2} = P(y_{0j}) = p_j, \quad j = 1,2, \quad (2.37)$$

where

$$P(y_{0j}) = 3(6-2c_1-c_2)+2(2a_1+a_2)y_{0j} - a_3 y_{0j}^2, \quad j=1,2, \tag{2.38}$$

and $p_j \in \mathbb{Z}-\{0\}$. In order to have a principal branch, at least one of the p_j should be a positive integer. Equation (2.4.b) gives

$$a_3 = -\frac{6-2c_1-c_2}{y_{01}y_{02}}, \quad 2a_1+a_2 = a_3(y_{01}+y_{02}). \tag{2.39}$$

Then (2.38) can be written as

$$P(y_{01}) = (6-2c_1-c_2)\left(1-\frac{y_{01}}{y_{02}}\right), \quad P(y_{02}) = (6-2c_1-c_2)\left(1-\frac{y_{02}}{y_{01}}\right). \tag{2.40}$$

If $p_1 p_2 \neq 0$ and $6-2c_1-c_2 \neq 0$, then p_j satisfy the following hyperbolic type of Diophantine equation

$$\frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{6-2c_1-c_2}. \tag{2.41}$$

For each solution set (p_1,p_2) of (2.41) one should find (r_{j1},r_{j2}) such that r_{ji}, $i=1,2$ are distinct integers and $r_{j1}r_{j2}=p_j$. Then y_{0j} and a_i can be obtained from (2.39), (2.40) and $r_{j1}+r_{j2} = a_1 y_{0j} - c_1 + 7$.

When $(c_1,c_2) = (3, -2+2/n^2)$, the Diophantine equation (2.41) takes the form

$$\frac{1}{p_1}+\frac{1}{p_2} = \frac{n^2}{2(n^2-1)}, \quad n \neq \pm 1. \tag{2.42}$$

The general solution of (2.42) is

$$p_1 = \frac{2(n^2-1)+d_i}{n^2}, \quad p_2 = \frac{2(n^2-1)}{n^2}\left[1+\frac{2(n^2-1)}{d_i}\right], \; n\neq 0, \tag{2.43}$$

where $\{d_i\}$ is the set of divisors of $4(n^2-1)^2 \neq 0$. When $n=\pm 3$, (2.43) gives $(p_1,p_2)=(2,16)$, which does not lead any Fuchs indices. $(p_1,p_2) = (1,-3)$, $(2,6)$, $(3,3)$, when $n=\pm 2$. We have distinct Fuchs indices for both branches only for $(p_1,p_2) = (2,6)$, $(3,3)$. If $(p_1,p_2)=(2,6)$, we have

$$y_{01} = -\frac{1}{a_1} : (r_{11},r_{12}) = (1,2),$$

$$y_{02} = \frac{3}{a_1} : (r_{21},r_{22}) = (1,6), \tag{2.44.a,b,c}$$

$$y''' = 3\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2} + a_1\left[yy''-(y')^2+\frac{1}{2}a_1 y^2 y'\right].$$

(2.44.c) does not pass the Painlevé test since the compatibility condition at $r_{12}=2$ is not satisfied identically.

If $(p_1,p_2)=(3,3)$, the Fuchs indices and the simplified equation are

$$y_{01}^2 = \frac{3}{2a_3}, y_{02} = -y_{01} : \; (r_{j1},r_{j2}) = (1,3), \; j=1,2,$$

$$y''' = 3\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2} + a_3 y^2 y'. \tag{2.45.a,b}$$

(2.45.c) was also considered in [12]. Integration of (2.45.c) once yields,

$$y'' = \frac{1}{2}\frac{(y')^2}{y} + a_3 y^3 + k_1 y^2, \quad k_1 = \text{constant}. \tag{2.46}$$

which is of Painlevé type [14].

After adding the non-dominant terms F_2 given in (2.2.b) to (2.45), the leading order $\tilde{\alpha}$ of u as $z\to z_0$ is $\tilde{\alpha}=-1$ and $(\tilde{r}_{j1},\tilde{r}_{j2}) = (1,3)$, $j=1,2$, if $A_{12}\neq 0$, $A_5=A_6=A_9=A_{15}=0$ and $A_{12}(z_0)u_0^2 = 3/2$. Then, we have the equation

$$y^2 y''' = 3yy'y'' - \frac{3}{2}(y')^3 + \frac{3}{2}y^4 y' + k_1 y' + k_2 y^2 y', \tag{2.47}$$

where k_1, k_2 are constants. Integration of (2.47) once yields

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + k_3 y^2 - k_2 y - \frac{k_1}{3y}, \; k_3 = \text{constant}. \tag{2.48}$$

(2.48) is of Painlevé type [14].

When $(c_1,c_2) = (3-1/n, -2+1/n+1/n^2)$, the general solution of the Diophantine equation (2.41) is

$$p_1 = \frac{2n^2+n-1+d_i}{n^2}, \quad p_2 = \frac{2n^2+n-1}{n^2}\left[1+\frac{2n^2+n-1}{d_i}\right], \; n\neq 0, \tag{2.49}$$

where $\{d_i\}$ is the set of divisors of $(2n^2+n-1)^2 \neq 0$. When $n=1$, $(p_1,p_2) = (1,-2), (3,6), (4,4)$. Only the solutions $(3,6)$ and $(4,4)$ give distinct Fuchs indices for both branches. The Fuchs indices and the simplified equations for these cases are as follows:

For $(p_1, p_2) = (3,6)$,

$$y_{01} = -\frac{1}{a_1}: \quad (r_{11}, r_{12}) = (1,3),$$

$$y_{02} = \frac{2}{a_1}: \quad (r_{21}, r_{22}) = (1,6), \qquad (2.50.a,b,c)$$

$$y''' = 2\frac{y'y''}{y} + a_1\left[yy'' - (y')^2 + a_1 y^2 y'\right].$$

(2.50.c) does not pass the Painlevé test, since the compatibility condition at $r_{12} = 3$ is not satisfied identically.

For $(p_1, p_2) = (4,4)$,

$$y_{01}^2 = \frac{2}{a_3}, y_{02} = -y_{01}: \quad (r_{j1}, r_{j2}) = (1,4), \quad j = 1,2,$$

$$y''' = 2\frac{y'y''}{y} + a_3 y^2 y'. \qquad (2.51.a,b)$$

(2.51.b) was also considered in [12]. Integrating (2.51.b) once yields,

$$y'' = a_3 y^3 + k_1 y^2, \quad k_1 = \text{constant}. \qquad (2.52)$$

(2.52) is of Painlevé type [14].

If we add the non dominant terms, then the leading order of u as $z \to z_0$ is $\tilde{\alpha} = -1$ and $(\tilde{r}_1, \tilde{r}_2) = (0,3)$ when $A_5 = 0$. The canonical form of the equation is as follows:

$$yy''' = 2y'y'' + 2y^3y' + k_1 yy', \; k_1 = \text{constant}. \quad (2.53)$$

(2.53) was also given in [11]. Integration of (2.53) once gives

$$y'' = 2y^3 + k_2 y^2 - \frac{k_1}{2}, \; k_2 = \text{constant}. \qquad (2.54)$$

(2.54) is solvable by means of the elliptic functions.

When $n = \pm 2, \pm 3$, the solutions of the Diophantine equation (2.49) do not give any Fuchs indices.

When $(c_1, c_2) = (3 - 3/n, -2 + 3/n - 1/n^2)$, the general solution of the Diophantine equation (2.41) is

$$p_1 = \frac{2n^2 + 3n + 1 + d_i}{n^2},$$

$$p_2 = \frac{2n^2 + 3n + 1}{n^2}\left[1 + \frac{2n^2 + 3n + 1}{d_i}\right], \; n \neq 0, \quad (2.55)$$

where $\{d_i\}$ is the set of divisors of $(2n^2 + 3n + 1)^2 \neq 0$. It should be noted that, $c_1 = c_2 = 0$ when $n = 1$. For $n = 2$ we have $(p_1, p_2) = (3, -15)$, $(4, 60)$, $(5, 15)$, $(6, 10)$, but only $(p_1, p_2) = (3, -15)$ gives the distinct Fuchs indices for both branches.

The Fuchs indices and the simplified equation of this case are

$$y_{01} = -\frac{3}{2a_1}: \quad (r_{11}, r_{12}) = (1,3),$$

$$y_{02} = -\frac{15}{4a_1}: \quad (r_{21}, r_{22}) = (-5,3), \qquad (2.56)$$

$$y''' = \frac{3}{2}\frac{y'y''}{y} - \frac{3}{4}\frac{(y')^3}{y^2} + a_1\left[yy'' + (y')^2\right] - \frac{1}{3}a_1^2 y^2 y'.$$

Without loss of generality, one can set $a_1 = 3/2$, then integrating (2.56) once yields

$$y'' = \frac{3}{4}\frac{(y')^2}{y} + \frac{3}{2}yy' - \frac{1}{4}y^3 + k_1, \qquad (2.57)$$

$$k_1 = \text{constant}.$$

This case was also given in [12], and (2.57) is of Painlevé type [3, 14].

If one adds the non-dominant terms, then $\tilde{\alpha} = -2$ and $(\tilde{r}_1, \tilde{r}_2) = (0,1)$. The transformation (2.5), the compatibility conditions at $(r_{11}, r_{12}) = (1,3)$, $r_{22} = 3$ and the compatibility conditions at $\tilde{r}_2 = 1$ allow one to determine all the coefficients A_k. Hence,

$$y^2 y''' = \frac{3}{2}yy'y'' - \frac{3}{4}(y')^3 - \frac{3}{2}\left[y^3 y'' + y^2 (y')^2\right]$$
$$- \frac{3}{4}y^4 y' + A_7 y^2 y' + A_7' y^3, \qquad (2.58)$$

where A_7 is an arbitrary analytic function of z. Integration of (2.58) once yields

$$y'' = \frac{3}{4}\frac{(y')^2}{y} - \frac{3}{2}yy' - \frac{1}{4}y^3 + A_7 y + k_1, \qquad (2.59)$$

where k_1 is an integration constant. (2.59) possesses the Painlevé property [3, 14].

For $n = -3, -2$, $(p_1, p_2) = (1, -10)$ and $(p_1, p_2) = (1,3)$, respectively. But for both cases there are double Fuchs index at ± 1. For $n = 3$, the only solution of (2.55) is $(p_1, p_2) = (4, 14)$. This solution gives the Fuchs indices $(r_{11}, r_{12}) = (1,4)$ for the first branch, but no Fuchs indices for the second branch.

When $(c_1, c_2) = (3 - 2/n, -2 + 2/n)$, the general solution of Diophantine equation (2.41) is

$$p_1 = \frac{2(n+1) + d_i}{n},$$
$$p_2 = \frac{2(n+1)}{n}\left[1 + \frac{2(n+1)}{d_i}\right],\ n \neq 0, \qquad (2.60)$$

where $\{d_i\}$ is the set of divisors of $4(n+1)^2 \neq 0$. $(p_1, p_2) = (2, -2(n+1))$ is a particular solution of the Diophantine equation which corresponds to $d_i = 2$. The Fuchs indices and the simplified equation corresponding to this case are as follows:

$$y_{01} = -\frac{n+2}{na_1} :\ (r_{11}, r_{12}) = (1, 2),$$
$$y_{02} = -\frac{(n+1)(n+2)}{na_1} :\ (r_{21}, r_{22}) = (-(1+n), 2),$$
$$y''' = \left(3 - \frac{2}{n}\right)\frac{y'y''}{y} - \left(2 - \frac{2}{n}\right)\frac{(y')^3}{y^2} \qquad (2.61.\mathrm{a,b})$$
$$+ a_1\left[yy'' - \frac{2n}{(n+2)^2}a_1y^2y'\right], n \neq 0, -1, -3$$

Without loss of generality, we can set $a_1 = 1 + 2/n$. If one lets $y = -u'/u$, and then $u' = v^n$, the last equation (2.61.c) yields

$$vv''' = v'v''. \qquad (2.62)$$

Integrating (2.62) once gives a linear equation for v. Therefore (2.61) is of Painlevé type and was also considered in [12].

In particular, for $n = -2$, (2.60) implies that $(p_1, p_2) = (2, 2)$. Then y_{0j}, the Fuchs indices for both branches and the simplified equation are as follows [12]:

$$y_{01}^2 = \frac{1}{a_3},$$
$$y_{02} = -y_{01} : (r_{j1}, r_{j2}) = (1, 2),\ j = 1, 2, \qquad (2.63)$$
$$y''' = 4\frac{y'y''}{y} - 3\frac{(y')^3}{y^2} + a_3y^2y'.$$

Integrating (2.63) once yields

$$y'' = \frac{(y')^2}{y} + a_3y^3 + k_1y^2,\ k_1 = \text{constant}. \qquad (2.64)$$

(2.64) is of Painlevé type [14].

After adding the non-dominant terms, one finds the following canonical form of the equations. If $A_5 \neq 0$, $A_6 = -3A_5$ and $A_9 = A_{12} = A_{15} = 0$, then $\tilde{\alpha} = -1$, $A_5(z_0)u_0 = -1$ and $(\tilde{r}_1, \tilde{r}_2) = (1, 3)$. The canonical form of the equation is

$$y^2y''' = 4yy'y'' - 3(y')^3 + y^4y' + A_5\left[yy'' - 3(y')^2 + y^4\right]$$
$$+ 3A_5'yy' - A_5''y^2, \qquad (2.65)$$

where A_5 is a locally analytic arbitrary function of z. If $A_{12} \neq 0$, and $A_5 = A_6 = A_9 = A_{15} = 0$, then $\tilde{\alpha} = -1$, $A_{12}(z_0)u_0^2 = 3$ and $(\tilde{r}_{j1}, \tilde{r}_{j2}) = (2, 3)$, $j = 1, 2$. The compatibility conditions at the Fuchs indices give

$$A_{10}'' - \frac{3}{2}\frac{A_{12}'}{A_{12}}A_{10}' + \frac{1}{2}\left(\frac{A_{12}'}{A_{12}}\right)^2 A_{10} = 0,$$
$$A_{11} = -\frac{3}{4}\frac{1}{A_{12}}A_{10}A_{10}' + \frac{3}{8}\frac{A_{12}'}{A_{12}}A_{10}^2,$$
$$A_{12}A_{12}'' = (A_{12}')^2, \quad A_{14} = -\frac{1}{3}A_{12}',$$
$$A_{13} = -A_{10}' + \frac{A_{12}'}{4A_{12}}A_{10}. \qquad (2.66)$$

Therefore, if $A_{12} = k_1 = \text{constant} \neq 0$, then the canonical form of the equation is

$$y^2y''' = 4yy'y'' - 3(y')^3 + y^4y' + (k_2 + k_3z)yy' + k_1y'$$
$$- \frac{3}{4}\frac{k_3}{k_1}(k_2 + k_3z)y^3 - k_3y^2. \qquad (2.67)$$

where k_2 and k_3 are constants. If $A_{12} = k_2e^{k_1z}$, $k_1k_2 \neq 0$, then the canonical form of the equation is

$$y^2y''' = 4yy'y'' - 3(y')^3 + y^4y' + \left(k_3e^{k_1z} + k_4e^{k_1z/2}\right)yy'$$
$$+ k_2e^{k_1z}y' - \frac{3}{8}\frac{k_1k_3}{k_2}\left(k_3e^{k_1z} + k_4e^{k_1z/2}\right)y^3$$
$$- \frac{1}{4}k_1\left(3k_3e^{k_1z} + k_4e^{k_12z/2}\right)y^2 - \frac{1}{3}k_1k_2e^{k_1z}y, \qquad (2.68)$$

where $k_i = \text{constant}$, $i = 1, \ldots, 4$. If $A_5 \neq 0$, $A_9 = A_{15} = 0$, $A_6 = -2A_5$ and $A_{12} = -A_5/2$, then $\tilde{\alpha} = -1$ and $A_5(z_0)u_{01} = -2 : (\tilde{r}_{11}, \tilde{r}_{12}) = (1, 2)$, $A_5(z_0)u_{02} = -6 :$

$(\tilde{r}_{21}, \tilde{r}_{22}) = (-3, 2)$. The canonical form of the equation in this case is

$$^2y''' = 4yy'y'' - 3(y')^3 + y^4y' + A_5\left[yy'' - 2(y')^2\right] \quad (2.69)$$
$$+ \frac{3}{2}A_5'yy' + A_{11}y^3 - \frac{1}{4}A_5^2y' - \frac{1}{2}A_5''y^2 + \frac{1}{4}A_5A_5'y,$$

where A_5, A_{11} are arbitrary locally analytic functions of z. Similarly, for $n = 1$ one can obtains the following canonical form of the equations, such that the corresponding simplified equation is not contained in (2.61.c):

$$yy''' = y'y'' + 4y^3y' + k_1y^2y'$$
$$- \left(\frac{k_1k_2}{6}z - k_3\right)y' + k_2y^2 + \frac{k_1k_2}{6}, \quad (2.70)$$

$$yy''' = y'y'' + 4y^3y' + \frac{1}{z}\left(yy'' - 2y^4\right) + \frac{k_1}{z}y^2y' - \frac{2k_1}{z^2}y^3$$
$$- \left(\frac{k_1^2}{2z^3} - \frac{k_2}{z}\right)y^2 + \left(\frac{k_1}{3z^3} - \frac{k_1^3}{108z^3} + \frac{k_1k_2}{6z} + k_3\right)y'$$
$$+ \left(\frac{4k_1}{3z^4} - \frac{k_1^3}{27z^4} + \frac{k_1k_2}{3z^2} + \frac{k_3}{z}\right)y, \quad (2.71)$$

$$yy''' = y'y'' + 4y^3y' - \frac{1}{z}\Big[3yy'' - 2(y')^2 - k_1z^2y^2y'$$
$$- 4y^4 - \frac{8}{3}k_1zy^3\Big] + \left(\frac{k_1^2}{2}z + \frac{k_2}{z}\right)y^2 \quad (2.72)$$
$$- \left(\frac{k_1^3}{144}z^3 + \frac{k_1k_2}{12}z - \frac{k_3}{z}\right)y' + \left(\frac{k_1^3}{36}z^2 + \frac{k_1k_2}{6}\right)y,$$

when $(A_1, A_2) = (0, 0)$, $(A_1, A_2) = (1/z, 0)$ and $(A_1, A_2) = ((A_2' - A_2^2)/A_2, 1/z)$, respectively, where k_i are constants. Integration of (2.70) and (2.71) yields

$$y'' = 2y^3 + k_1y^2 + (k_2z + k_4)y + \frac{k_1k_2}{6}z + k_3, \quad (2.73)$$

$$v'' = 2v^3 + (k_4z - k_2)v - (k_3 + \frac{k_1k_4}{6}), \quad (2.74)$$

respectively, where k_4 is an integration constant and $v = y + (k_1/6z)$ in (2.74).

When $(c_1, c_2) = (3, -2)$, the solutions of the Diophantine equation (2.41) are $(p_1, p_2) = (1, -2), (3, 4), (4, 6)$. $(1, -2)$ gives a double Fuchs index and the others do not lead to any Fuchs indices.

Case III. $a_4 \neq 0$: In this case there are three branches corresponding to roots $y_{0j},\ j = 1, 2, 3$, of (2.4.b). Equation (2.4.b) implies that

$$\prod_{j=1}^{3} y_{0j} = -\frac{6 - 2c_1 - c_2}{a_4},$$

$$\sum_{i\neq j}^{3} y_{0i}y_{0j} = \frac{1}{a_4}(2a_1 + a_2), \quad \sum_{j=1}^{3} y_{0j} = \frac{a_3}{a_4}. \quad (2.75)$$

Let

$$P(y_{0j}) = 3(6 - 2c_1 - c_2) + 2(2a_1 + a_2)y_{0j}$$
$$- a_3y_{0j}^2,\ j = 1, 2, 3. \quad (2.76)$$

If the Fuchs indices (except $r_{j0} = -1$) are $r_{ji},\ i = 1, 2$, corresponding to y_{0j}, then (2.4.a) implies that

$$\prod_{i=1}^{2} r_{ji} = P(y_{0j}) = p_j. \quad (2.77)$$

In order to have a principal branch, p_j should be integers such that at least one of them is positive. Equations (2.75) and (2.76) give

$$p_j = (6 - 2c_1 - c_2)\prod_{l=1,\ l\neq j}^{3}\left(1 - \frac{y_{0j}}{y_{0l}}\right),\ \ j = 1, 2, 3, \quad (2.78)$$

and hence p_j satisfies the Diophantine equation

$$\sum_{j=1}^{3}\frac{1}{p_j} = \frac{1}{6 - 2c_1 - c_2}, \quad (2.79)$$

If $\prod_{j=1}^{3} p_j \neq 0$ and $6 - 2c_1 - c_2 \neq 0$; from (2.78) one has the following system for y_{0j}:

$$p_1(y_{02} - y_{03}) = \mu y_{01},$$
$$p_2(y_{03} - y_{01}) = \mu y_{02}, \quad (2.80)$$
$$p_3(y_{01} - y_{02}) = \mu y_{03},$$

where

$$\mu = \frac{6 - 2c_1 - c_2}{y_{01}y_{02}y_{03}} \cdot (y_{01} - y_{02})(y_{02} - y_{03})(y_{01} - y_{03}). \quad (2.81)$$

On the other hand, (2.78) gives that

$$\prod_{j=1}^{3} p_j = -(6-2c_1-c_2)\mu^2. \tag{2.82}$$

Then, if $a_1 \neq 0$ (note that $r_{j1}+r_{j2} = a_1 y_{0j} - c_1 + 7$) then $(6-2c_1-c_2)\mu^2 > 0$ and a real number. Therefore, $\prod_{j=1}^{3} p_j < 0$. That is, if $p_1 > 0$, then either p_2 or p_3 is a negative integer. So one should consider cases $a_1 = 0$ and $a_1 \neq 0$ separately.

III.A. $a_1 = 0$: From (2.4.a), one has

$$r_{j1}+r_{j2} = 7-c_1. \tag{2.83}$$

Thus c_1 is an integer. Since

$$(r_{j1}-r_{j2})^2 = (r_{j1}+r_{j2})^2 - 4r_{j1}r_{j2}, \tag{2.84}$$

$(7-c_1)^2 - 4p_j$ is a perfect square. Then for each five cases one can determine p_j. By using the system (2.80) and (2.75), one obtains y_{0j} and a_m, $m = 2,3,4$.

When $(c_1, c_2) = (3, -2+\frac{2}{n^2})$, since $c_1 = 3$, (2.84) and (2.83) give

$$(r_{j1}+r_{j1})^2 = 16-4p_j, \quad j = 1,2,3. \tag{2.85}$$

So $16-4p_2$ must be a perfect square. If we let $p_1, p_2 > 0$, then (2.85) implies that $p_1 = p_2 = 3$. The Diophantine equation (2.79) implies that p_3 is an integer when $n = \pm 1$. But $6-2c_1-c_2 = 0$ when $n = \pm 1$.

When $(c_1, c_2) = (3-\frac{1}{n}, -2+\frac{1}{n}+\frac{1}{n^2})$, c_1 is an integer and $6-2c_1-c_2 \neq 0$ only if $n = 1$. The Fuchs indices and the simplified equation for this case are [12]

$$y_{0j}^3 = -\frac{2}{a_4}:\ (r_{j1}, r_{j2}) = (2,3),\ j = 1,2,3,$$

$$y''' = 2\frac{y'y''}{y} + a_4 y^4. \tag{2.86.a,b}$$

If we add the non-dominant terms to (2.86), then $\tilde{\alpha} = -1$, $u_0 =$ arbitrary $\neq 0$ and the Fuchs indices are $(\tilde{r}_1, \tilde{r}_2) = (0,3)$. The transformation (2.5), the compatibility conditions at (r_{j1}, r_{j2}), $j = 1,2,3$, and at $(\tilde{r}_1, \tilde{r}_2)$ imply that $A_k = 0$, $k = 1,\ldots,11$. So the canonical form of the equation is the simplified equation (2.86.b).

When $(c_1, c_2) = (3-\frac{3}{n}, -2+\frac{3}{n}-\frac{1}{n^2})$, $c_1 \in \mathbb{Z}$ implies that $n = \pm 1, \pm 3$. But only for $n = -3$, $6-2c_1-c_2 \neq 0$, $c_1^2+c_2^2 \neq 0$ and the Fuchs indices are distinct for all three branches. The indices and the simplified equation for this case are

$$y_{01} = -\frac{1}{3a_2}:\ (r_{11}, r_{12}) = (1,2),$$

$$y_{02} = \frac{2}{3a_2}:\ (r_{21}, r_{22}) = (1,2),$$

$$y_{03} = \frac{5}{3a_2}:\ (r_{31}, r_{32}) = (-2,5), \tag{2.87.a – d}$$

$$y''' = 4\frac{y'y''}{y} - \frac{28}{9}\frac{(y')^3}{y^2} + a_2\left[(y')^2 + 6a_2 y^2 y' + 3a_2^2 y^4\right].$$

(2.87.d) does not pass the Painlevé test, since the compatibility conditions are not satisfied identically.

When $(c_1, c_2) = (3-\frac{2}{n}, -2+\frac{2}{n})$, $c_1 \in \mathbb{Z}$ implies that $n = \pm 1, \pm 2$. For these values of n, there are no distinct Fuchs indices for all branches.

When $(c_1, c_2) = (3, -2)$, the solutions of the Diophantine equation (2.79) do not give distinct Fuchs indices.

III.B. $a_1 \neq 0$: Once the solution set $p_j = r_{j1}r_{j2}$, $j = 1,2,3$, of (2.79) is known, y_{0j} and a_i, $i = 1,2,3,4$, can be determined from the equations (2.80), (2.75) and

$$r_{j1}+r_{j2} = a_1 y_{0j} + 7 - c_1,\ j = 1,2,3. \tag{2.88}$$

When $(c_1, c_2) = (3, -2+\frac{2}{n^2})$, $(p_1, p_2, p_3) = (2, 4(n-1), -4(n+1))$ is a particular solution of the Diophantine equation (2.79), and $\mu = \pm 4n$. The Fuchs indices are distinct only for $\mu = -4n$. The indices and the simplified equation for this case are

$$y_{01} = -\frac{1}{a_1}:\ (r_{11}, r_{12}) = (1,2),$$

$$y_{02} = \frac{n-1}{a_1}:\ (r_{21}, r_{22}) = (4, n-1),$$

$$y_{03} = -\frac{n+1}{a_1}:\ (r_{31}, r_{32} = (4, -(n+1)),$$

$$y''' = 3\frac{y'y''}{y} - \frac{2(n^2-1)}{n^2}\frac{(y')^3}{y^2} + a_1\left[yy'' - \frac{6}{n^2}(y')^2 + \frac{6}{n^2}a_1 y^2 y' - \frac{2}{n^2}a_1^2 y^4\right],$$

$$n \neq 0, \pm 1, \pm 5. \tag{2.89.a – d}$$

(2.89.d) was also considered in [12]. If one lets $y = u'/u$ and $u' = v^n$ then (2.89.d) yields

$$vv''' = 3v'v''. \tag{2.90}$$

Integrating (2.90) once gives $v'' = k_1 v^3$, $k_1 =$ constant. If $k_1 = 0$, then $v = k_2 z + k_3$, $k_i =$ constant, $i = 2,3$. If

$k_1 \neq 0$, then $v = \sum_{i=0}^{\infty} v_{4i}(z-z_0)^{4i-1}$, where z_0 is arbitrary. Since $u' = v^n$, in order to u, and consequently y, being single valued, it is necessary and sufficient that u' does not contain the term $(z-z_0)^{-1}$. That is $n \neq 0, \pm(1+4m)$, where $m \in \mathbb{Z}_+$.

Particularly for $n = 2$, after adding the non-dominant terms to (2.89), $\tilde{\alpha} = -1$ is the possible leading order of $u = 1/y$ as $z \to z_0$ if $A_{12} \neq 0$, $A_5 = A_6 = A_9 = A_{15} = 0$ and $A_{12}(z_0)u_0^2 = 3/2$. The Fuchs indices are $(\tilde{r}_{j1}, \tilde{r}_{j2}) = (1,3)$, $j = 1,2$ and the canonical form of the equation is

$$y^2y''' = 3yy'y'' - \frac{3}{2}(y')^3 - y^3y'' + \frac{3}{2}y^2(y')^2 + \frac{3}{2}y^4y' + \frac{1}{2}y^6 + \left(\frac{k_1}{3}z^2 + k_2z + k_3\right)(y^2y' + y^4) - \left(\frac{2k_1}{3}z + k_2\right)y^3 + k_1y' + \frac{k_1}{3}y, \tag{2.91}$$

where $k_i =$ constant, $i = 1,2,3$.

When $(c_1, c_2) = (3 - \frac{1}{n}, -2 + \frac{1}{n} + \frac{1}{n^2})$, $(p_1, p_2, p_3) = (2, 6(2n-1), -3(n+1))$ is a particular solution of (2.79). Then the system (2.80) has non-trivial solution if $\mu = \pm 6n$. For both values of μ, we have the following simplified equations:

$$y_{01} = -\frac{n+1}{na_1}: \ (r_{11}, r_{12}) = (1,2),$$

$$y_{02} = -\frac{(n+1)^2}{na_1}: \ (r_{21}, r_{22}) = (3, -(n+1)),$$

$$y_{03} = \frac{(n+1)(2n-1)}{na_1}: \ (r_{31}, r_{32}) = (6, 2n-1)$$

$$y''' = \left(3 - \frac{1}{n}\right)\frac{y'y''}{y} - \left(2 - \frac{1}{n} - \frac{1}{n^2}\right)\frac{(y')^3}{y^2} + a_1\left[yy'' - \frac{3}{n(n+1)}(y')^2 + \frac{3-n}{(n+1)^2}a_1y^2y' - \frac{n}{(n+1)^3}a_1^2y^4\right], \quad n \neq 0, -1, -4. \tag{2.92.a – d}$$

(2.92.d) was also considered in [12]. Substitution of $y = u'/u$ in (2.92) and then letting $u' = v^n$ give the following equation for v

$$vv''' = 2v'v'' \tag{2.93}$$

Integration of (2.93) once gives $v'' = k_1v^2$, $k_1 =$ constant. If $k_1 = 0$ then $v = k_2z + k_3$, $k_i =$ constants $i = 2,3$. If $k_1 \neq 0$, then $v = \sum_{i=0}^{\infty} v_{6i}(z-z_0)^{6i-2}$, $z_0 =$ arbitrary. Therefore, if $n \neq -3m-1$, $m = 0,1,2,\ldots$, u and consequently y is a single valued function of z.

In particular for $n = 1$, $(p_1, p_2, p_3) = (3, 5, -30)$, $(2, N, -N)$, $N \in \mathbb{Z}_+$ exist solutions of (2.79). For $(p_1, p_2, p_3) = (3, 5, -30)$, the system (2.80) has a non-trivial solution if $\mu = \pm 15$. Only the $\mu = -15$ case gives distinct Fuchs indices for all branches. The simplified equations for this case are as follows:

$$y_{01} = -\frac{1}{a_1}: \ (r_{11}, r_{12}) = (1,3),$$

$$y_{02} = \frac{1}{a_1}: \ (r_{21}, r_{22}) = (1,5),$$

$$y_{03} = -\frac{4}{a_1}: \ (r_{31}, r_{32}) = (-5,6), \tag{2.94.a – d}$$

$$y''' = 2\frac{y'y''}{y} + a_1\left[yy'' - \frac{3}{2}(y')^2 + 2a_1y^2y' - \frac{1}{2}a_1^2y^4\right].$$

Letting $a_1 = -1$, integrating (2.94.d) once yields

$$y'' = \frac{3}{2}\frac{(y')^2}{y} + \frac{1}{2}y^3 + k_1, \ k_1 = \text{constant}, \tag{2.95}$$

which is solvable by means of elliptic functions [14]. After adding the non dominant terms $\tilde{\alpha} = -1$ if $A_5 = 0$, the indices are $(\tilde{r}_1, \tilde{r}_2) = (0, 3)$. Then the canonical form of the equation is

$$yy''' = 2y'y'' - y^2y'' + \frac{3}{2}y(y')^2 + 2y^3y' + \frac{1}{2}y^5 + k_1y, \quad k_1 = \text{constant}. \tag{2.96}$$

For $(p_1, p_2, p_3) = (2, N, -N)$, (2.80) implies that $\mu = \pm N$. For $\mu = N$, $y_{01} = 0$, and $\mu = -N$ we have the following equation:

$$y''' = 2\frac{y'y''}{y} + a_1\left[yy'' + \frac{N^2+12}{4-N^2}(y')^2 - \frac{16}{4-N^2}a_1y^2y' + \frac{4}{4-N^2}a_1^2y^4\right], \ N \neq \pm 2, \tag{2.97}$$

with $a_1y_{01} = -2$, $a_1y_{02} = (N-2)/2$, $a_1y_{03} = -(N+2)/2$, and $(r_{11}, r_{12}) = (1,2)$. The Fuchs indices for the second and the third branches satisfy the following equations, respectively,

$$r_{2i}^2 - \frac{N+8}{2}r_{2i} + N = 0, \qquad r_{3i}^2 + \frac{8-N}{2}r_{3i} - N = 0 \tag{2.98}$$

The compatibility condition at $r_{12}=2$ is not satisfied identically unless $N=6$. Then from (2.98), the indices are $(r_{21},r_{22})=(1,6)$ and $(r_{31},r_{32})=(-2,3)$. The corresponding simplified equation is (2.92) for $n=1$. If one adds the non-dominant terms then, $(\tilde r_1,\tilde r_2)=(0,3)$. The transformation (2.5), the compatibility conditions at $(r_{11},r_{12})=(1,2)$, $(r_{21},r_{22})=(1,6)$, $r_{32}=3$ of y and the compatibility conditions at the Fuchs indices of u imply that all the coefficients A_k are zero except $A_{10}=k_1=$ constant. So, the canonical form of the equation is

$$yy'''=2y'y''-2y^2y''+3y(y')^2+2y^3y'+y^5+k_1y. \tag{2.99}$$

When $(c_1,c_2)=(3-3/n,-2+3/n-1/n^2)$, $(p_1,p_2,p_3)=(2,2(2n+1),-(n+1))$ is a particular solution of (2.79). For these values of p_j the system (2.80) has a nontrivial solution if $\mu=\pm 2n$. Only for $\mu=2n$ there are distinct indices for all three branches. The indices and the corresponding simplified equation are

$$y_{01}=-\frac{n+3}{na_1}:\ (r_{11},r_{12})=(1,2), \qquad (2.100.\text{a}-\text{d})$$

$$y_{02}=-\frac{(n+3)(2n+1)}{na_1}:\ (r_{21},r_{22})=(-(2n+1),-2),$$

$$y_{03}=-\frac{(n+3)(n+1)}{na_1}:(r_{31},r_{32})=(-(n+1),1),$$

$$\begin{aligned}y'''&=3\Big(1-\frac{1}{n}\Big)\frac{y'y''}{y}-\Big(2-\frac{3}{n}+\frac{1}{n^2}\Big)\frac{(y')^3}{y^2}\\&\quad+a_1\Big[yy''+\frac{3}{n(n+3)}(y')^2-\frac{3(n+1)}{(n+3)^2}\,a_1y^2y'\\&\qquad+\frac{n}{(n+3)^3}\,a_1^2y^4\Big],\ n\neq -1,-2,-3.\end{aligned}$$

(2.100.d) was also considered in [12]. Substituting $y=u'/u$ in (2.100) and letting $u'=v^n$ gives

$$v'''=0. \tag{2.101}$$

(2.101) has the solution $v(z)=k_1z^2+k_2z+k_3,\ k_i=$ constant. Therefore, the zeros z_0 of v are singularities of u' when $n<0$. Hence, it is necessary and sufficient that $n>0$, that u' does not contain the term $(z-z_0)^{-1}$. Then movable singularities of u and consequently y are poles only.

If we let $n=2$ and add the non-dominant terms, then $\tilde\alpha=-1$ and $(\tilde r_1,\tilde r_2)=(0,1)$. The canonical form of the equation is

$$\begin{aligned}y^2y'''&=\frac{3}{2}yy'y''-\frac{3}{4}(y')^3-\frac{5}{2}y^3y''-\frac{3}{4}y^2(y')^2-\frac{9}{4}y^4y'\\&\quad-\frac{1}{4}y^6+A_7(y^2y'+y^4)+A_{11}y^3,\end{aligned} \tag{2.102}$$

where A_7,A_{11} are arbitrary locally analytic functions of z.

When $(c_1,c_2)=(3-2/n,-2+2/n)$, and $n=1$ the solutions of the Diophantine equation (2.79) are $(p_1,p_2,p_3)=(3,24,-8)$, $(3,132,-11)$, $(5,16,-80)$, $(5,19,-380)$, $(6,10,-60)$, $(7,8,-56)$, $(4,-N,N)$, $N\in\mathbb{Z}_+$. Only for $(3,24,-8)$ and $(4,-N,N)$ there are distinct Fuchs indices for all branches. The indices and the simplified equations for these cases are as follows:

For $(p_1,p_2,p_3)=(3,24,-8)$:

$$y_{01}=-\frac{2}{a_1}:\ (r_{11},r_{12})=(1,3),$$

$$y_{02}=\frac{4}{a_1}:\ (r_{21},r_{22})=(4,6),$$

$$y_{03}=-\frac{4}{a_1}:\ (r_{31},r_{32})=(-2,4), \qquad (2.103.\text{a}-\text{d})$$

$$y'''=\frac{y'y''}{y}+a_1\Big(yy''+\frac{1}{4}a_1y^2y'-\frac{1}{8}a_1^2y^4\Big).$$

This case was considered in [12]. After adding the non-dominant terms, if $A_5=0$, then $\tilde p=-1$ and the Fuchs indices are $(\tilde r_1,\tilde r_2)=(0,2)$. The transformation (2.5) and the compatibility conditions at $(r_{11},r_{12})=(1,3)$, $(r_{21},r_{22})=(4,6)$ and at $\tilde r_2=2$ imply that $A_m=0,\ m=1,2\ldots,6$ and

$$\begin{aligned}&A_7^{(4)}+A_7A_7''+(A_7'-k_1)(A_7'+2k_1)=0,\\&A_8=A_7'+k_1,\quad A_9=k_1,\quad A_{10}=-A_8',\end{aligned} \tag{2.104}$$

where k_1 is a constant. It should be noted that the equation for A_7 is the autonomous part of the second member of the first Painlevé hierarchy [6,8]. From (2.104) we have the following two cases: If $k_1=0$ and $A_7=-12/z^2$ then

$$yy'''=y'y''-2y^2y''+y^3y'+y^5-\frac{12}{z^2}y^3+\frac{24}{z^3}y'+\frac{72}{z^4}y. \tag{2.105}$$

If $A_7 = k_2 z + k_3$, $k_i =$ constant, $i = 2,3$, then the canonical form of the equation is

$$yy''' = y'y'' - 2y^2y'' + y^3y' + y^5 + (k_2z + k_3)y^3 + k_2(2y' + y^2). \quad (2.106)$$

For $(p_1, p_2, p_3) = (4, -N, N)$: $p_1 = 4$, implies that $(r_{11}, r_{12}) = (1,4)$ and hence $a_1y_{01} = -1$. By using the system (2.80), one finds y_{02} and y_{03} in terms of a_1 and N. So, the Fuchs indices r_{2i} and r_{3i}, $i = 1,2$, satisfy the equations

$$r_{2i}^2 - \frac{44+N}{8} r_{2i} + N = 0, \qquad r_{3i}^2 - \frac{44-N}{8} r_{3i} - N = 0, \quad (2.107)$$

respectively, and the simplified equation is

$$y''' = \frac{y'y''}{y} + a_1yy'' - 2\frac{N^2-144}{16-N^2} a_1(y')^2 - \frac{512}{16-N^2} a^2y^2y' + \frac{256}{16-N^2} a_1^3y^4, \; N \neq \pm 4. \quad (2.108)$$

The compatibility condition at $r_{12} = 4$ is not satisfied identically unless $N = 12$. Then, (2.107) gives that $(r_{21}, r_{22}) = (3,4)$ and $(r_{31}, r_{32}) = (-2,6)$, respectively. Thus we have the simplified equation [12]

$$y''' = \frac{y'y''}{y} + a_1(yy'' + 4a_1y^2y' - 2a_1^2y^4). \quad (2.109)$$

For this case, the canonical form of the equation is $\tilde{\alpha} = -1$, $(\tilde{r}_1, \tilde{r}_2) = (0,2)$ and

$$yy''' = y'y'' - y^2y'' + 4y^3y' + 2y^5 + (2k_1z + k_2)y^3 + k_1(y' + y^2), \quad (2.110)$$

where k_1, k_2 are constants.

When $(c_1, c_2) = (3, -2)$, the solutions of the Diophantine equation (2.79) do not lead to any distinct Fuchs indices.

3. Leading Order α = –2

$\alpha = -2$ is also a possible leading order of the equation (1.4). By adding the term yy', the following simplified equation with the leading order $\alpha = -2$, is obtained

$$y''' = c_1\frac{y'y''}{y} + c_2\frac{(y')^3}{y^2} + ayy', \quad (3.1)$$

where a is constant and c_1, c_2 are given by (1.12), (1.15) and (1.16).

Substituting $y = y_0(z-z_0)^{-2} + \beta(z-z_0)^{r-2}$ into (3.1) gives the following equations for the Fuchs indices r and y_0, respectively:

$$Q(r) = (r+1)[r^2 + 2(c_1-5)r + 24 - 12c_1 - 8c_2] = 0, \qquad ay_0 = 12 - 6c_1 - 4c_2. \quad (3.2.a,b)$$

(3.2.b) implies that there is only one branch. In order to have a principal branch, the indices r_1 and r_2 (except $r_0 = -1$) should be distinct positive integers. Then (3.2.a) implies that $2c_1$ and $4(3c_1 + 2c_2)$ should be integers.

To find the canonical forms of the equations, one should consider the following equations for $c_2 = 0$ and $c_2 \neq 0$, respectively:

$$yy''' = c_1y'y'' + ay^2y'' + A_1yy'' + A_2(y')^2 + A_3y^3 + A_4yy' + A_5y'' + A_6y^2 + A_7y' + A_8y + A_9, \quad (3.3)$$

$$y^2y''' = c_1yy'y'' + c_2(y')^3 + ay^3y' + A_1y^2y'' + A_2y(y')^2 + A_3y^4 + A_4y^2y' + A_5yy'' + A_6(y')^2 + A_7y^3 + A_8yy' + A_9y'' + A_{10}y^2 + A_{11}y' + A_{12}y + A_{13}, \quad (3.4)$$

where the A_k are locally analytic functions of z. The coefficients A_k can be found by using the procedure described in the previous section.

When $(c_1, c_2) = (3, -2 + \frac{2}{n^2})$, the Fuchs indices satisfy $r_1 + r_2 = 4$ and $r_1r_2 = 4[1 - (4/n^2)]$. Hence, $n = \pm1, \pm2, \pm4$, but $n = \pm1$ does not lead a principal branch. Therefore we have the following cases: For $n = \pm2$, the Fuchs indices, simplified equation and the canonical form of the equation are

$$y_0 = \text{arbitrary}: \; (r_1, r_2) = (0, 4), \qquad y''' = 3\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2}, \quad (3.5)$$

$$y''' = 3\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2} + \frac{1}{y^2}[(k_1z + k_3)y' + k_2], \quad (3.6)$$

respectively, where k_i, $i = 1,2,3$ are constants.

For $n = \pm4$:

$$ay_0 = \frac{3}{2}: \quad (r_1, r_2) = (1, 3), \qquad y''' = 3\frac{y'y''}{y} - \frac{15}{8}\frac{(y')^3}{y^2} + ayy', \quad (3.7.a,b)$$

Integration of (3.7.b) once yields

$$v'' = \frac{1}{2}\frac{(v')^2}{v} + av^3 + k_1 v^2, \quad k_1 = \text{constant}, \tag{3.8}$$

where $v^2 = y$. If we let $a = 3/2$, then (3.8) is of Painlevé type [14]. For this case, the canonical form of the equation is

$$y''' = 3\frac{y'y''}{y} - \frac{15}{8}\frac{(y')^3}{y^2} + \frac{3}{2}yy' + k_1\frac{y'}{y}, \quad k_1 = \text{constant}. \tag{3.9}$$

Integration of (3.9) yields

$$v'' = \frac{1}{2}\frac{(v')^2}{v} + 3v^2 - \frac{2k_1}{3}\frac{1}{v} + \frac{k_2}{2}v^2, \tag{3.10}$$

where $v^2 = y$ and k_2 is an integration constant. (3.10) is solvable by means of the elliptic functions [14].

When $(c_1, c_2) = (3 - 1/n, -2 + 1/n + 1/n^2)$, $2c_1$ is an integer if $n = \pm 1, \pm 2$. $n = -1$ does not lead a principal branch. So, when $n = -2$, we have the following simplified equation

$$y_0 = \text{arbitrary}: \ (r_1, r_2) = (0, 3), \qquad y''' = \frac{7}{2}\frac{y'y''}{y} - \frac{9}{4}\frac{(y')^3}{y^2}, \tag{3.11}$$

and the canonical form of the equation

$$y''' = \frac{7}{2}\frac{y'y''}{y} - \frac{9}{4}\frac{(y')^3}{y^2} + k_1\frac{y'}{y}, \tag{3.12}$$

where k_1 is a constant. (3.12) yields

$$u'' = \frac{5}{4}\frac{(u')^2}{u} + \frac{k_1}{2}u^2 + k_2, \ k_2 = \text{constant} \tag{3.13}$$

after letting $y = 1/u$ and integrating once. (3.13) is of Painlevé type [14]. When $n = 1$, we have

$$y_0 = \text{arbitrary}: \ (r_1, r_2) = (0, 6), \qquad y''' = 2\frac{y'y''}{y}. \tag{3.14}$$

The canonical form of the equations are as follows:

$$y''' = 2\frac{y'y''}{y} + k_1, \quad k_1 = \text{constant} \tag{3.15}$$

$$y''' = 2\frac{y'y''}{y} + (k_2 - 2k_1 z)\frac{y'}{y} + k_1, \quad k_1, k_2 = \text{constant} \tag{3.16}$$

For $n = 2$, the simplified equation is

$$y_0 = \frac{2}{a}: \ (r_1, r_2) = (1, 4), \qquad y''' = \frac{5}{2}\frac{y'y''}{y} - \frac{5}{4}\frac{(y')^3}{y^2} + ayy'. \tag{3.17.a,b}$$

Integration of (3.17.b) once yields

$$y'' = \frac{5}{4}\frac{(y')^2}{y} + \frac{a}{2}y^2 + k_1, \quad k_1 = \text{constant}. \tag{3.18}$$

(3.18) is solvable by means of elliptic functions. The canonical form of the equation is

$$y''' = \frac{5}{2}\frac{y'y''}{y} - \frac{5}{4}\frac{(y')^3}{y^2} + 2yy' + k_1 y', \quad k_1 = \text{constant}. \tag{3.19}$$

Integration of (3.19) yields

$$v'' = \frac{3}{2}\frac{(v')^2}{v} + \frac{1}{2}v^3 - \frac{k_1}{2}v + \frac{k_2}{2}\frac{1}{v}, \tag{3.20}$$

where $v^2 = y$ and k_2 is an integration constant. (3.20) is solvable by means of elliptic functions.

When $(c_1, c_2) = (3 - \frac{3}{n}, -2 + 3/n - 1/n^2)$, $2c_1 =$ integer implies that $n = \pm 1, \pm 2, \pm 3, \pm 6$. If $n = 1, -1$ and $n = \pm 3, \pm 6$ then $c_1 = c_2 = 0$, there is no principal branch and there are no Fuchs indices respectively. Therefore, we have the following cases: For $n = -2$ the simplified equation is

$$y_0 = \text{arbitrary}: \ (r_1, r_2) = (0, 1), \qquad y''' = \frac{9}{2}\frac{y'y''}{y} - \frac{15}{4}\frac{(y')^3}{y^2}, \tag{3.21}$$

and the canonical form of the equation is

$$y''' = \frac{9}{2}\frac{y'y''}{y} - \frac{15}{4}\frac{(y')^3}{y^2} + k_1 y' + k_2\frac{y'}{y}, \tag{3.22}$$

where k_i, $i = 1, 2$ are constants. If we let $y = 1/u$ in (3.22) and integrate once, then we have

$$u'' = \frac{3}{4}\frac{(u')^2}{u} + \frac{k_2}{2}u^2 + k_1 u + k_3, \tag{3.23}$$

where k_3 is an integration constant and (3.23) is of Painlevé type [14]. For $n = 2$, the simplified equation is

$$y_0 = \frac{6}{a}: \ (r_1, r_2) = (3, 4), \qquad y''' = \frac{3}{2}\frac{y'y''}{y} - \frac{3}{4}\frac{(y')^3}{y^2} + ayy'. \tag{3.24.a,b}$$

Letting $a = 6$ and integrating (3.24.b) once yields

$$y'' = \frac{3}{4}\frac{(y')^2}{y} + 3y^2 + k_1, \; k_1 = \text{constant.} \tag{3.25}$$

(3.25) is of Painlevé type [3, 14]. The canonical form is

$$y''' = \frac{3}{2}\frac{y'y''}{y} - \frac{3}{4}\frac{(y')^3}{y^2} + 6yy' + (k_1 z + k_2)y' + 2k_1 y, \tag{3.26}$$

where $k_i =$ constant, $i = 1, 2$.

When $(c_1, c_2) = (3 - 2/n, -2 + 2/n)$, $2c_1$ is integer if $n = \pm 1, \pm 2, \pm 4$. When $n = -1$, there is no principal branch. So, we have the following three cases:

For $n = -4$,

$$y_0 = \frac{1}{a} : \; (r_1, r_2) = (1, 2), \qquad y''' = \frac{7}{2}\frac{y'y''}{y} - \frac{5}{2}\frac{(y')^3}{y^2} + ayy'. \tag{3.27.a,b}$$

Setting $a = 1$ in (3.27.b) and integrating once yields

$$v'' = \frac{(v')^2}{v} + v^3 + k_1 v^2, \; k_1 = \text{constant,} \tag{3.28}$$

where $v^2 = y$. (3.28) is of Painlevé type [14]. The canonical form is

$$y''' = \frac{7}{2}\frac{y'y''}{y} - \frac{5}{2}\frac{(y')^3}{y^2} + yy' + k_1\frac{y'}{y}. \tag{3.29}$$

Integration of (3.29) once yields

$$v'' = \frac{(v')^2}{v} + v^3 - \frac{k_1}{3v} + \frac{k_2}{2}v^2, \tag{3.30}$$

where $v^2 = y$ and $k_i, \; i = 1, 2$, are constants. (3.30) is solvable by means of elliptic functions. For $n = -2$, we have the following simplified equation and the canonical form of the equation:

$$y_0 = \text{arbitrary} : \; (r_1, r_2) = (0, 2), \qquad y''' = 4\frac{y'y''}{y} - 3\frac{(y')^3}{y^2}, \tag{3.31}$$

$$y''' = 4\frac{y'y''}{y} - 3\frac{(y')^3}{y^2} + k_1\frac{y'}{y}, \; k_1 = \text{constant.} \tag{3.32}$$

respectively. Integrating (3.32) once gives

$$y'' = \frac{(y')^2}{y} + k_2 y^2 - \frac{k_1}{2}, . \tag{3.33}$$

where k_2 is an integration constant. For $n = 1$, the simplified equation is

$$y_0 = \frac{6}{a} : \quad (r_1, r_2) = (2, 6), \qquad y''' = \frac{y'y''}{y} + ayy'. \tag{3.34.a,b}$$

Integration of (3.34.b) once yields

$$y'' = ay^2 + k_1 y, \; k_1 = \text{constant.} \tag{3.35}$$

(3.35) is solvable by means of elliptic functions. If $A_1 = A_2 = 0$, the canonical form of the equations is

$$y''' = \frac{y'y''}{y} + 6yy' - \Big(\frac{1}{24}k_1^2 z^2 + k_2 z + k_3\Big)\frac{y'}{y} + k_1 y + \Big(\frac{1}{12}k_1^2 z + k_2\Big), \tag{3.36}$$

where $k_i, \; i = 1, 2, 3$, are constants. Integration of (3.36) once yields

$$y'' = 6y^2 + (k_1 z + k_4)y + \frac{1}{24}k_1^2 z^2 + k_2 z - k_3, \quad k_4 = \text{constant.} \tag{3.37}$$

If one lets $y = v - (k_1 z + k_4)/12$ in (3.37), then it yields the first Painlevé equation. If $A_1 = 1/2z, \; A_2 = 0$, then the canonical form of thek equation is

$$y''' = \frac{y'y''}{y} + 6yy' + \frac{1}{2z}\left(y'' - 6y^2\right) + \frac{5}{8}\frac{y}{z^3} - \Big(\frac{639}{5120}\frac{1}{z^4} - k_1 z - k_2\Big)\frac{y'}{y} - \Big(\frac{5751}{1280}\frac{1}{z^5} - \frac{k_2}{2z} + \frac{k_1}{2}\Big). \tag{3.38}$$

For $n = 2$, the simplified equation is

$$y_0 = \frac{4}{a} : \; (r_1, r_2) = (2, 4), \qquad y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + ayy'. \tag{3.39.a,b}$$

Integration of (3.39.b) once yields

$$y'' = \frac{(y')^2}{y} - \frac{a}{2}y^2 + k_1, \; k_1 = \text{constant.} \tag{3.40}$$

(3.40) is of Painlevé type [14]. To obtain the canonical forms we have two possibilities, depending on the leading order $\tilde{\alpha}$. If $A_{11} \neq 0$ and $A_5 = A_6 = A_9 = A_{13} = 0$, then $\tilde{\alpha} = -1$ and $A_{11}(z_0)u_0^2 = 1$,

$(\tilde{r}_{j1},\tilde{r}_{j2}) = (1,2)$, $j = 1,2$. The compatibility conditions at $(r_1,r_2) = (2,4)$ and at $\tilde{r}_{ij}$ give that $A_1' + A_1^2 = 0$. Therefore, we have

$$y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + 4yy' + k_1\frac{y'}{y^2} + k_2 y 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + 4yy' + \frac{1}{z}\left(y'' - \frac{1}{2}\frac{(y')^2}{y} - 4y^2\right) - k_2\frac{y}{z} + k_1\frac{y'}{y^2} + \frac{k_1}{2zy}, \tag{3.41}$$

for $A_1 = 0$ and $A_1 = 1/z$, respectively, where k_1, k_2 are constants. $\tilde{\alpha} = -2$ is also a leading order, and the Fuchs indices are $(\tilde{r}_1,\tilde{r}_2) = (0,2)$. The canonical equation is

$$y''' = 2\frac{y'y''}{y} - \frac{(y')^3}{y^2} + 4yy' + k_1\left[y'' - \frac{(y')^2}{y} - 2y^2\right] + k_2 y. \tag{3.42}$$

For $n = 4$, we have the following simplified equation

$$y_0 = \frac{3}{a}: \ (r_1,r_2) = (2,3), \qquad y''' = \frac{5}{2}\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2} + ayy'. \tag{3.43.a,b}$$

Setting $a = 3$ and integrating (3.43.b) once yields

$$v'' = \frac{(v')^2}{v} + v^3 + k_1, \quad k_1 = \text{constant}, \tag{3.44}$$

where $v^2 = y$. (3.44) is of Painlevé type [14]. If $A_8 \neq 0$, then $\tilde{\alpha} = -2$ and $A_8(z_0)u_0 = 1$, $(\tilde{r}_1,\tilde{r}_2) = (1,2)$ and the canonical form is

$$y''' = \frac{5}{2}\frac{y'y''}{y} - \frac{3}{2}\frac{(y')^3}{y^2} + 3yy' + \frac{1}{6}\frac{A_8'}{A_8}\left[y'' - \frac{3}{2}\frac{(y')^2}{y}\right] + A_8\frac{y'}{y} - \frac{4}{3}A_8', \tag{3.45}$$

where A_8 satisfies

$$A_8 A_8'' = \frac{2}{3}(A_8')^2. \tag{3.46}$$

If $A_8 = k_1 = $ constant, integration of (3.45) yields

$$v'' = \frac{(v')^2}{v} + v^3 - 2k_1\frac{1}{v} + k_2, \tag{3.47}$$

where $v^2 = y$ and k_2 is an integration constant. (3.47) is solvable by means of elliptic functions.

When $(c_1, c_2) = (3, -2)$, this case does not lead to any distinct Fuchs indices.

4. Leading Order α = –3

$\alpha = -3$ is also a possible leading order of equation (1.4). By adding the term y^2, the following simplified equation with the leading order $\alpha = -3$, is obtained:

$$y''' = c_1\frac{y'y''}{y} + c_2\frac{(y')^3}{y^2} + ay^2, \tag{4.1}$$

where a is constant and c_1, c_2 are given by (1.12), (1.15) and (1.16). In this case the Fuchs indices r and y_0 satisfy the equations

$$Q(r) = (r+1)[r^2 - (13-3c_1)r + 60 - 36c_1 - 27c_2] = 0, \qquad ay_0 = -60 + 36c_1 + 27c_2, \tag{4.2a,b}$$

respectively. (4.2.b) implies that there is only one branch. In order to have positive distinct Fuchs indices, $3c_1$ and $36c_1 + 27c_2$ both must be integers for all five cases.

To find the canonical forms of the equations, one should consider the following equations for $c_2 = 0$ and $c_2 \neq 0$

$$yy''' = c_1 y'y'' + ay^3 + A_1 yy'' + A_2(y')^2 + A_3 yy' + A_4 y^2 + A_5 y'' + A_6 y' + A_7 y + A_8, \tag{4.3}$$

$$y^2 y''' = c_1 yy'y'' + c_2(y')^3 + ay^4 + A_1 y^2 y'' + A_2 y(y')^2 + A_3 y^2 y' + A_4 y^3 + A_5 yy'' + A_6(y')^2 + A_7 yy' + A_8 y^2 + A_9 y'' + A_{10} y' + A_{11} y + A_{12}, \tag{4.4}$$

where the A_k are locally analytic functions of z. The coefficients A_k can be found by using the same procedure described in the previous sections.

When $(c_1, c_2) = (3, -2 + 2/n^2)$, n takes the values of $\pm 1, \pm 3$. But $n = \pm 1$ does not lead a principal branch. For $n = \pm 3$ we have the following simplified equation and the canonical form of the equation

$$y_0 = \text{arbitrary}: \ (r_1,r_2) = (0,4), \qquad y''' = 3\frac{y'y''}{y} - \frac{16}{9}\frac{(y')^3}{y^2}, \tag{4.5}$$

$$y''' = 3\frac{y'y''}{y} - \frac{16}{9}\frac{(y')^3}{y^2} + (k_1 z + k_2)y' + k_2 y, \quad k_1, k_2 = \text{constant}. \tag{4.6}$$

When $(c_1, c_2) = (3-1/n, -2+1/n+1/n^2)$, n takes the values of $n = \pm 1, \pm 3$. There is no principal branch and there are no Fuchs indices for $n = -1$ and $n = 1$, respectively. Hence we have the following cases: For $n = -3$, the simplified equation and the canonical equation are

$$\begin{aligned} &y_0 = \text{arbitrary}: \ (r_1, r_2) = (0,3), \\ &y''' = \frac{10}{3}\frac{y'y''}{y} - \frac{20}{9}\frac{(y')^3}{y^2}, \end{aligned} \tag{4.7}$$

$$y''' = \frac{10}{3}\frac{y'y''}{y} - \frac{20}{9}\frac{(y')^3}{y^2} + k_1, \ \ k_1 = \text{constant}. \tag{4.8}$$

respectively. For $n = 3$, the simplified equation and the canonical equation are

$$\begin{aligned} &y_0 = -\frac{6}{a}: \quad (r_1, r_2) = (2,3), \\ &y''' = \frac{8}{3}\frac{y'y''}{y} - \frac{14}{9}\frac{(y')^3}{y^2} + ay^2. \end{aligned} \tag{4.9}$$

$$\begin{aligned} y''' = {} & \frac{8}{3}\frac{y'y''}{y} - \frac{14}{9}\frac{(y')^3}{y^2} - 6y^2 \\ & + k_1\frac{y''}{y} - \frac{3k_1}{2}\frac{1}{y^2}\left[(y')^2 - k_1 y' + \frac{k_1^2}{4}\right], \end{aligned} \tag{4.10}$$

respectively, where $k_1 = \text{constant}$.

When $(c_1, c_2) = (3-3/n, -2+3/n-1/n^2)$, n takes the values $\pm 1, \pm 2, \pm 9$. But, we have the principal branch only for $n = -3$, and the simplified equation and the canonical equation are:

$$y_0 = \text{arbitrary}: \ (r_1, r_2) = (0,1),$$

$$y''' = 4\frac{y'y''}{y} - \frac{28}{9}\frac{(y')^3}{y^2}, \tag{4.11}$$

$$y''' = 4\frac{y'y''}{y} - \frac{28}{9}\frac{(y')^3}{y^2} + A_3 y' + A_4 y, \tag{4.12}$$

respectively, where A_3 and A_4 are arbitrary locally analytic functions of z.

When $(c_1, c_2) = (3-2/n, -2+2/n)$, n takes the values of $\pm 1, \pm 2, \pm 3$. But, we have only the following simplified equation and the canonical form which corresponds to $n = 1$:

$$y_0 = 1: \ (r_1, r_2) = (4,6), \ y''' = \frac{y'y''}{y} - 24y^2, \tag{4.13}$$

$$y''' = \frac{y'y''}{y} - 24y^2 + k_1 y + \left(\frac{k_1^2}{12}z + k_2\right)\frac{y'}{y} - \frac{k_1^2}{12}, \tag{4.14}$$

respectively, where k_1, k_2 are constants.

When $(c_1, c_2) = (3, -2)$, this case does not lead any distinct Fuchs indices.

In conclusion, we obtained the canonical forms of non-polynomial third order equations with the leading orders $\alpha = -1, -2, -3$, such that all pass the Painlevé test. Not the canonical forms, but the simplified equations, except (2.17), (2.26) and (2.94) given in section 2, were considered in the literature before [11, 12]. The simplified equations given in section 2 can be obtained by differentiating the leading terms of the third Painlevé equation and adding the terms of order -4 as $z \to z_0$ with constant coefficients, such that $y = 0, \infty$ are the only singular values of the equation in y, and they are of the order ε^{-3} or greater, if one lets $z = z_0 + \varepsilon t$. Hence, these equations can be considered as the generalization of the third Painlevé equation.

In the third and fourth sections, we investigated the cases of leading order $\alpha = -2, -3$, which were not considered before. We found that 20 new canonical forms of non-polynomial third order equations, namely (3.6), (3.9), (3.12), (3.15), (3.16), (3.19), (3.22), (3.26), (3.29), (3.32), (3.36), (3.38), (3.41), (3.42), (3.45), (4.6), (4.8), (4.10), (4.12), and (4.14), all pass the Painlevé test.

In the procedure we imposed the existence of at least one principal branch. i.e. the resonances are distinct positive integers for one branch. But the compatibility conditions at positive resonances for the second and third branches are identically satisfied for each case. Instead of having positive distinct integer resonances, one can consider the case of distinct negative integer resonances. In this case it is possible to obtain equations belonging to Chazy classes.

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